Universality and universal finite-size scaling functions in four-dimensional Ising spin glasses

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We study the four-dimensional Ising spin glass with Gaussian and bond-diluted bimodal distributed interactions via large-scale Monte Carlo simulations and show via an extensive finite-size scaling analysis that four-dimensional Ising spin glasses obey universality.

I. INTRODUCTION

The concept of universality, according to which the values of different quantities, such as for example critical exponents, do not depend on the microscopic details of a model, is well established in the theory of critical phenomena of systems without frustration. However, for spin glasses,[1] which have both frustration and disorder, the situation has been less clear until recently. Large-scale simulations in three space dimensions[2,3,4] for different disorder distributions of the random interactions between the spins have shown that the critical exponents, the values of different observables at criticality, as well as the finite-size scaling functions—all which are necessary ingredients to show that different models are in the same universality class—agree well within error bars. The conclusion that universality holds has also been obtained via other methods such as high-temperature series expansions[5] where the critical exponent γ has been studied. Studies of dynamical quantities however have yielded different critical exponents for different disorder distributions,[6,7,8,9] although it is unclear up to what level it can be expected that "dynamical universality" can be compared to universality in thermal equilibrium.

In two space dimensions things are less clear: Because the transition to a spin-glass phase only happens at zero temperature, it is believed that systems with discrete and continuous coupling distributions behave in a different way.[10] In particular, the ground-state entropy is nonzero in the former while it is zero in the latter. Thus one might expect that at the zero-temperature critical point the critical exponent η is different. Recently, however, an alternate scenario for universality in two space dimensions has been proposed,[11] where it is expected that as long as the temperature is nonzero two-dimensional spin glasses with different disorder distributions belong to the same universality class and where the different behavior seen at zero temperature is explained by the presence of an additional spurious fixed-point.[12] Limitations in the simulation techniques and analysis methods have so far yielded no conclusive results making this proposal controversial.[13,14]

In spin glasses it is extremely difficult and numerically very costly to determine critical exponents with high precision. This is mainly due to the following reason: It is difficult to sample the disorder average with good enough statistics, especially for large system sizes, because spin glasses have very long equilibration times in Monte Carlo simulations and as a consequence one has to deal with corrections to scaling due to a very limited range of system sizes at hand. In previous studies[2,3] only statistical error bars had been considered. Because of limited system sizes and corrections to scaling in three space dimensions, deviations between the critical parameters beyond statistical error bars can be expected and indeed this expectation has very recently been confirmed in Ref. 4 in a very thorough study where for the first time corrections to scaling have been studied with good accuracy. While studying higher-dimensional systems than three space dimensions might seem paradoxical at first because of the aforementioned problems, the proximity to the upper critical dimension $d_{\rm ucd} = 6$ is advantageous. High-temperature series expansion studies[5] suggest that corrections to scaling should be falling off fast in four-dimensional Ising spin glasses, i.e., that the leading correction-to-scaling exponent ω is large. More specifically, in Ref. 5 a value for ω between 1.3 and 1.6 was found. Although the range of system sizes accessible to Monte Carlo simulations in four space dimensions is more limited than in three space dimensions, the model poses a "good compromise" case where corrections can be kept small while the system sizes are reasonably large. In this article we choose the observables that display the smallest corrections to scaling for the system sizes accessible via Monte Carlo simulations. We feel that introducing correction-to-scaling terms along the lines of Ref. 15 in order to give more precise estimates for the critical exponents cannot be controlled sufficiently well, and thus, we do not display an analysis using these methods. However, we show that our results are perfectly compatible with a large correction-to-scaling exponent ω.

As with the three-dimensional Ising spin glass,[3,16,17] there have been many different estimates for the critical exponents (especially for the anomalous dimension η) of

the four-dimensional Ising spin glass as shown in Table I.

The main conclusion of this work is that equilibrium universality in four-dimensional spin glasses is satisfied, since we find agreement within error bars for all the finite-size scaling functions and critical exponents studied.

In Sec. II we introduce the model as well as the measured observables. In addition, we describe the numerical methods used. Results are presented in Sec. III, followed by concluding remarks in Sec. IV. Finally, a discussion of other commonly-used observables that have been less useful in the present analysis is given in Appendix A. Details of the analysis are presented in Appendices B and C, as well as an extended scaling analysis of the data with Gaussian disorder in Appendix D.

II. MODEL, OBSERVABLES, AND NUMERICAL DETAILS

A. Edwards-Anderson model

The Edwards-Anderson (EA) Ising spin-glass Hamiltonian[1,28] is given by

$$\mathcal{H} = -\sum_{\langle i,j \rangle} J_{ij} S_i S_j. \tag{1}$$

The Ising spins $S_i = \pm 1$ lie on a hypercubic lattice of size $N = L^d$ in $d = 4$ space dimensions with periodic boundary conditions. The sum is over nearest neighbors on the lattice. We study two versions of the model:

(i) Gaussian-distributed interactions J_{ij} with zero mean and standard deviation unity;

$$\mathcal{P}(J_{ij}) = \frac{1}{\sqrt{2\pi}} e^{-J_{ij}^2/2}. \tag{2}$$

(ii) Bimodal-distributed random interactions[29] with a bond dilution of 65%, i.e.,

$$\begin{aligned} \mathcal{P}(J_{ij}) &= (1-p)\,\delta(J_{ij}) \\ &+ \frac{p}{2}\,[\delta(J_{ij}-1) + \delta(J_{ij}+1)], \end{aligned} \tag{3}$$

with $p = 0.35$ (Ref. 30).

B. Measured observables

In order to compute the critical parameters and henceforth test for universality, we compute different observables that are known to show a good signature of the phase transition. The Binder cumulant,[31] defined via

$$g(L,T) = \frac{1}{2}\left(3 - \frac{[\langle q^4 \rangle]_{\mathrm{av}}}{[\langle q^2 \rangle]_{\mathrm{av}}^2}\right), \tag{4}$$

is dimensionless, and scales as

$$g(L,T) = \widetilde{g}[BL^{1/\nu}(T-T_c)] + \text{corrections}. \tag{5}$$

Here T_c is the critical temperature and B is a metric factor. The critical exponent ν describes the divergence of the infinite-volume correlation length $\xi(T)$ as the temperature approaches T_c, i.e., $\xi(T) \sim |T-T_c|^{-\nu}$. The corrections to scaling in Eq. (5), as well as in Eqs. (9) and (12) are asymptotically dominated by the leading correction-to-scaling exponent ω and vanish in the thermodynamic limit ($L \to \infty$). In Eq. (4) $\langle \mathcal{O} \rangle$ represents a thermal average of an observable $\mathcal{O}$, $[\mathcal{O}]_{\mathrm{av}}$ is a disorder average, and q is the spin overlap;

$$q = \frac{1}{N} \sum_{i=1}^{N} S_i^a S_i^b. \tag{6}$$

In Eq. (6) "a" and "b" represent two replicas of the system with the same disorder. In addition, we study the finite-size correlation length,[32,33,34]

$$\xi(L,T) = \frac{1}{2\sin(|\mathbf{k}_{\min}|/2)}\left[\frac{\chi_{\mathrm{SG}}(\mathbf{0})}{\chi_{\mathrm{SG}}(\mathbf{k}_{\min})} - 1\right]^{1/2}, \tag{7}$$

where $\mathbf{k}_{\min} = (2\pi/L, 0, 0, 0)$ is the smallest nonzero wave vector and the wave-vector-dependent spin-glass susceptibility is given by

$$\chi_{\mathrm{SG}}(\mathbf{k}) = \frac{1}{N} \sum_{i,j} [\langle S_i S_j \rangle^2]_{\mathrm{av}}\, e^{\imath \mathbf{k} \cdot (\mathbf{R}_i - \mathbf{R}_j)}. \tag{8}$$

The finite-size correlation length $\xi(L,T)$ divided by the system size is also a dimensionless quantity, i.e.,

$$\frac{\xi(L,T)}{L} = \widetilde{\xi}[BL^{1/\nu}(T-T_c)] + \text{corrections}. \tag{9}$$

For $L \to \infty$ data for $\xi(L,T)/L$ as well as for $g(L,T)$ intersect at $T \to T_c$. For finite systems the data cross at an effective critical temperature $T_c^*(L)$ that converges asymptotically to T_c as[31]

$$T_c^*(L) - T_c \propto L^{-\omega-1/\nu}. \tag{10}$$

In the following we denote the value of an observable $\mathcal{O}$ measured at this effective critical point by

$$\mathcal{O}^* \doteq \mathcal{O}[T_c^*(L)]. \tag{11}$$

The definition of the finite-size correlation length $\xi(L,T)$ in Eq. (7) involves in general the same leading corrections to scaling as χ_{SG}, which in turn is given by ω. Furthermore, this definition is not unique[35] and different definitions of $\xi(L,T)/L$ show differences of the order of L^{-2}. Such differences may seem irrelevant, but for the small systems sizes that can be accessed in spin-glass simulations correction terms might actually be visible in the data. Note that in Eqs. (5), (9), and (12), T_c and

TABLE I: Different estimates (sorted chronologically and with respect to the method used for the determination) of the critical exponents computed by different groups for Gaussian (G), bimodal ($\pm J$), triangular (T), uniform (U) and Laplacian (L) random bonds. The estimates, especially the ones for η, show strong variations and often do not agree. The last two rows show the results from this study. The critical temperatures denoted by an asterisk indicate that the variance of the coupling distribution used in the corresponding study is not normalized to unity (Ref. 18).

Authors	Couplings	Method	γ	ν	η	T_c
Singh & Chakravarty (Ref. 19)	$\pm J$	Series	2.0(4)			2.02(6)
Klein *et al.* (Ref. 20)	$\pm J$		2.00(25)	0.95	-0.11	2.04(3)
Daboul *et al.* (Ref. 5)	U		2.4(1)			1.10(2)*
	$\pm J$		2.5(3)			1.96(7)
	G		2.3(1)			1.79(1)
	T		2.5(2)			1.36(3)*
Bernardi & Campbell (Ref. 21)	$\pm J$	Dynamic MC			$-0.31(1)$	1.99(1)
	G				$-0.47(2)$	1.77(1)
	U				$-0.37(2)$	1.91(1)
	L				$-0.60(3)$	1.52(1)
Bhatt & Young (Ref. 22)	G	Static MC	1.8(4)	0.8	$-0.30(15)$	1.75(5)
Reger *et al.* (Ref. 23)	$\pm J$				-0.5	
Parisi *et al.* (Ref. 24)	G		2.1(2)	0.9(1)	$-0.35(5)$	1.80(1)
Ney-Nifle (Ref. 25)	G			0.87(15)		1.80(5)
Marinari & Zuliani (Ref. 26)	$\pm J$			1.0(1)	$-0.30(5)$	2.03(3)
Hukushima (Ref. 27)	$\pm J$			0.92(6)		2.00(4)
This study	G	Static MC	2.32(8)	1.02(2)	$-0.275(25)$	1.805(10)
	$\pm J$ bond-diluted		2.33(6)	1.025(15)	$-0.275(25)$	1.0385(25)*

the metric factor B are nonuniversal, but, since B is included explicitly, the scaling functions $\widetilde{g}(x)$ and $\widetilde{\xi}(x)$ are both universal.[3,36] Since for both disorder distributions studied we use the same boundary conditions and sample shapes, these scaling functions are expected to be the same for different disorder distributions if the systems are in the same universality class. This is a necessary yet not sufficient condition. In addition, the critical exponents, as well as the values of the scaling functions at criticality ($T = T_c$) have to agree.

In addition to studying the Binder cumulant as well as the finite-size correlation length (from which we obtain T_c and the critical exponent ν), we need to study another observable to obtain a *second* critical exponent to fully characterize the universality class of the model.[37] Thus we also study the scaling behavior of the spin-glass susceptibility $\chi_{\rm SG} = \chi_{\rm SG}(\mathbf{k} = 0)$ (also $\chi_{\rm SG} = N[\langle q^2 \rangle]_{\rm av}$). The spin-glass susceptibility scales as

$$\chi_{\rm SG}(L,T) = CL^{2-\eta}\widetilde{\chi}[BL^{1/\nu}(T-T_c)] + \text{corrections}, \quad (12)$$

where the anomalous dimension η is the second critical exponent needed to establish the universality class of the model. In Eq. (12) C represents a nonuniversal amplitude. We also study another related quantity $\zeta(L,T)$ that has shown to be useful in determining the critical exponent η, which is defined as[38]

$$\zeta(L,T) = \frac{\chi_{\rm SG}}{\xi^2}. \quad (13)$$

The advantage of studying ζ is mainly given by the fact that the statistical correlations between $\chi_{\rm SG}$ and ξ lead to smaller errors on ζ than on $\chi_{\rm SG}$ and therefore to a more precise determination of η.

Finally, we study other phenomenological couplings that have been suggested to compute the spin-glass transition temperature, which are the lack of self-averaging A given by[39,40,41,42,43]

$$A(L,T) = \frac{[\langle q^2 \rangle^2]_{\rm av} - [\langle q^2 \rangle]_{\rm av}^2}{[\langle q^2 \rangle]_{\rm av}^2}, \quad (14)$$

and the Guerra parameter G given by[44,45]

$$G(L,T) = \frac{[\langle q^2 \rangle^2]_{\rm av} - [\langle q^2 \rangle]_{\rm av}^2}{[\langle q^4 \rangle]_{\rm av} - [\langle q^2 \rangle]_{\rm av}^2}. \quad (15)$$

Both A and G are related to the Binder cumulant g through the relation $g = 1 - A/(2G)$.

The scaling expressions in Eqs. (5), (9), and (12) can be used to determine the critical exponents, but in practice this strategy is not very promising because especially in the case of the spin-glass susceptibility analytic corrections to scaling are very easily confused with the leading scaling behavior, which in turn leads to unreliable estimates of the critical exponents.[3,4] In order to determine reliable estimates for the critical quantities, we use the quotient method, which avoids the problems of the analytic corrections to scaling in an elegant manner.[34,46,47] For any observable $\mathcal{O}(L,T)$ and the finite-size correlation length, finite-size scaling theory predicts[48,49,50] that

$$\frac{\mathcal{O}(L,T)}{\mathcal{O}(\infty,T)} = f_{\mathcal{O}}\Big[\xi(\infty,T)/L\Big] + O\Big(\xi^{-\omega}, L^{-\omega}\Big), \quad (16)$$

as well as

$$\frac{\mathcal{O}(sL,T)}{\mathcal{O}(L,T)} = F_{\mathcal{O}}\Big[\xi(L,T)/L; s\Big] + O\Big(\xi^{-\omega}, L^{-\omega}\Big), \quad (17)$$

where $f_{\mathcal{O}}$ and $F_{\mathcal{O}}$ are universal finite-size scaling functions and $s > 1$ is a scale factor. The exponent ω is again the leading nonanalytic correction-to-scaling exponent. Because in Eq. (17) only pairs of *finite* system sizes L and sL appear, this formulation is well adapted for use in numerical simulations. For example, the knowledge of the universal scaling functions F_{χ} and F_{ξ} [meaning $\mathcal{O} = \chi$ or $\mathcal{O} = \xi$ in Eq. (17), respectively] allows us to extract the critical exponents η and ν using the quotient method. For the quotient method one defines an effective critical temperature T_c^* at which the correlation length measured in units of the lattice size L is equal for the pair of systems, i.e.,

$$\xi(L, T_c^*)/L = \xi(sL, T_c^*)/(sL), \qquad (18)$$

or alternatively,

$$g(L, T_c^*) = g(sL, T_c^*). \qquad (19)$$

Note that we do not use a different notation for T_c^* defined through Eqs. (18) and (19) although the corresponding T_c^* in general is different, and only in the thermodynamic limit converges to a unique value T_c as indicated in Eq. (10). The crossings are determined by fitting a cubic spline through the data. We have refrained from using reweighting techniques for the determination of the crossings,[46,51] because the remaining statistical errors of the sample average dominate the errors even for the good statistics we have at hand. In addition to Eq. (17) we also study the following equivalent relation

$$\frac{\mathcal{O}(sL, T)}{\mathcal{O}(L, T)} = F_{\mathcal{O}}\Big[g(L, T); s\Big] + O\Big(\xi^{-\omega}, L^{-\omega}\Big) \qquad (20)$$

which, however, will prove to be advantageous in the present study. Another case in which this version of the finite-size scaling relation is very useful is to study scaling properties *within* the spin-glass phase.[52]

The critical exponent $x_{\mathcal{O}}$ associated with a given observable which at criticality diverges as $(T - T_c)^{-x_{\mathcal{O}}}$ can then be estimated via the quotient

$$s^{x_{\mathcal{O}}/\nu} = \frac{\mathcal{O}(sL, T_c^*)}{\mathcal{O}(L, T_c^*)} + O(L^{-\omega}), \qquad (21)$$

and thus from the finite-size scaling function $F_{\mathcal{O}}$. In Eq. (21) the critical exponent ν of the correlation length is unknown, but it can be estimated for example from the finite-size scaling function of the temperature derivative of the correlation length $\xi(L, T)$, $F_{\partial_T \xi}$, via

$$s^{1/\nu} = 1 + \frac{x^*}{s} \partial_x F_{\xi}(x, s)\big|_{x=x^*} + O(L^{-\omega}), \qquad (22)$$

with $x = \xi(L, T)/L$ and $x^* = \xi(L, T_c^*)/L$. Alternatively, ν can also be estimated from the temperature derivative of the finite-size scaling function of the Binder cumulant $F_{\partial_T g}$, i.e.,

$$s^{1/\nu} = 1 + g^* \partial_g F_g(g, s)\big|_{g=g^*} + O(L^{-\omega}). \qquad (23)$$

Here $g^* = g(T_c^*, L)$. In our study we fit cubic splines to the data of F_{ξ} and F_g to calculate the derivatives in Eqs. (22) and (23). A detailed derivation of Eqs. (22) and (23) is given in Appendix B.

The anomalous dimension η can be determined from F_{χ}. Using the scaling relation

$$\gamma = \nu(2 - \eta), \qquad (24)$$

we obtain

$$s^{\gamma/\nu} = s^{2-\eta} = F_{\chi}^* + O(L^{-\omega}). \qquad (25)$$

Similarly, η can be obtained from the finite-size scaling function F_{ζ},

$$s^{-\eta} = F_{\zeta}^* + O(L^{-\omega}). \qquad (26)$$

We determine the critical exponent γ from Eq. (24) using the estimates of ν and η. Finally, we also study the Binder cumulant as a function of $\xi(L, T)/L$ (Ref. 2,3, 4 and 53). Nonuniversal metric factors cancel and so $g(L, T) = \hat{g}[\xi(L, T)/L]$ with $\hat{g}$ a universal function.[2,3] Therefore data for different disorder distributions should fall on the same universal curve if the models share the same universality class.

C. Simulation details

The simulations are done using exchange (parallel tempering) Monte Carlo[54,55] and the simulation parameters are presented in Tables II and III for the Gaussian and bond-diluted bimodal disorder distributions, respectively. For the Gaussian disorder we test equilibration using the method presented in Ref. 56. For the bond-diluted bimodal disorder we use a multispin coded approach to speed up the simulation in addition to a cluster updating routine,[2,57] which can substantially speed up equilibration when the system is diluted and complements the parallel tempering Monte Carlo updates. Although we did no systematic study, we have the impression that in four dimensions the additional cluster updates are somewhat less effective than in lower space dimensions. Equilibration in the bond-diluted case is tested by a logarithmic binning of the data. Once the last three bins for all observables agree within error bars, we declare the system to be in thermal equilibrium.

III. RESULTS

In the following we perform a finite-size scaling analysis of different observables for both models. We determine quantities with small scaling corrections that provide good estimates for the critical temperature and critical exponents. We then compare different finite-size scaling functions for the model with different disorder distributions and discuss the influence of corrections to scaling on our results.
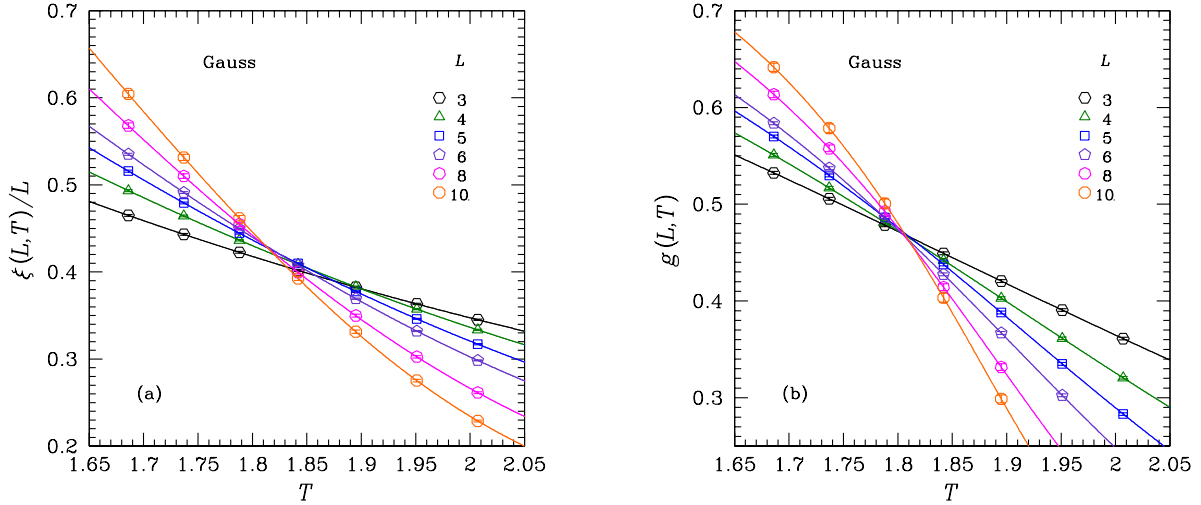
FIG. 1: (Color online) Determination of the critical temperature of the four-dimensional Edwards-Anderson Ising spin glass with Gaussian disorder. In panel (a) the finite-size correlation length $\xi(L,T)/L$ as a function of the temperature T for different system sizes L is shown. The data for $L \geq 6$ cross at $T_c = 1.810(15)$. In panel (b) the corresponding data for the Binder cumulant $g(L,T)$ as a function of the temperature T for different system sizes L are shown. The data cross at $T_c = 1.805(10)$, in agreement with the data for the correlation length. The crossing of the data for the Binder cumulant is cleaner than for the correlation length and shows no noticeable drift of the crossing point with increasing system sizes.
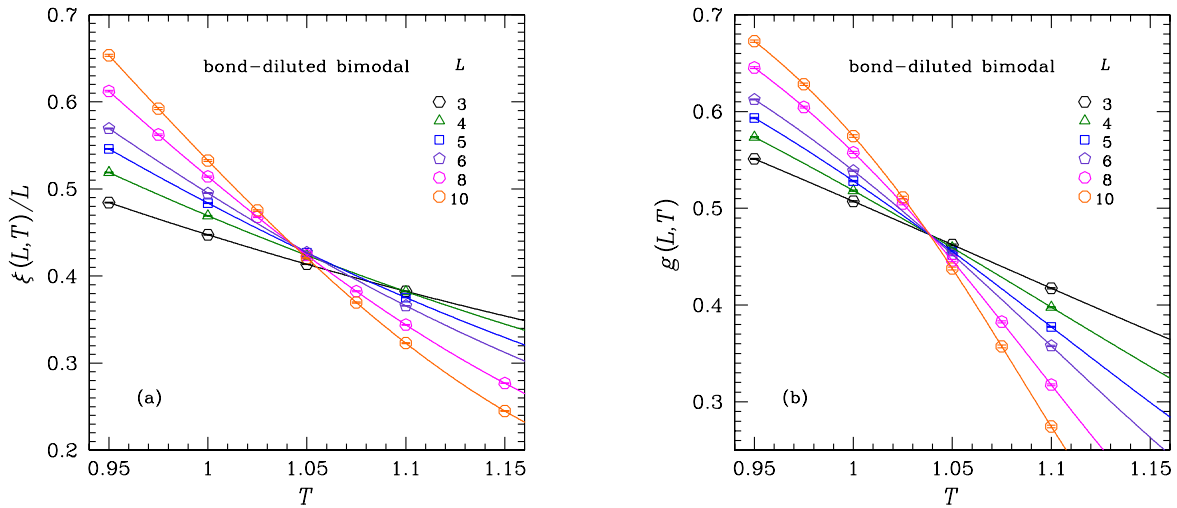

FIG. 2: (Color online) Determination of the critical temperature of the four-dimensional Edwards-Anderson Ising spin glass with diluted bimodal disorder. In panel (a) the finite-size correlation length $\xi(L,T)/L$ as a function of the temperature T for different system sizes L is shown. The data for $L \geq 6$ cross at $T_c = 1.042(5)$. In panel (b) the corresponding data for the Binder cumulant $g(L,T)$ as a function of the temperature T for different system sizes L are shown. The data cross cleanly at $T_c = 1.0385(25)$. The crossing of the data for the Binder cumulant is again more precise than the one of the correlation length and shows no noticeable drift of the crossing point with increasing system sizes.

A. Gaussian disorder (no dilution)

Panel (a) of Fig. 1 shows the data for the finite-size correlation length as a function of the temperature for different system sizes L. With increasing L, the data show a shift of the effective T_c toward a smaller value of T_c. This effect has already been observed in studies of the three-dimensional EA model,[2,3,58] however, in contrast to the three-dimensional case the range of available lattice sizes in four dimensions is more restricted and therefore the effect of this shift is clearly a restriction for a precise determination of T_c using the crossings of $\xi(L,T)/L$. Taking the data for $L \geq 6$ and neglecting the remaining scaling corrections, we obtain $T_c = 1.810(15)$.
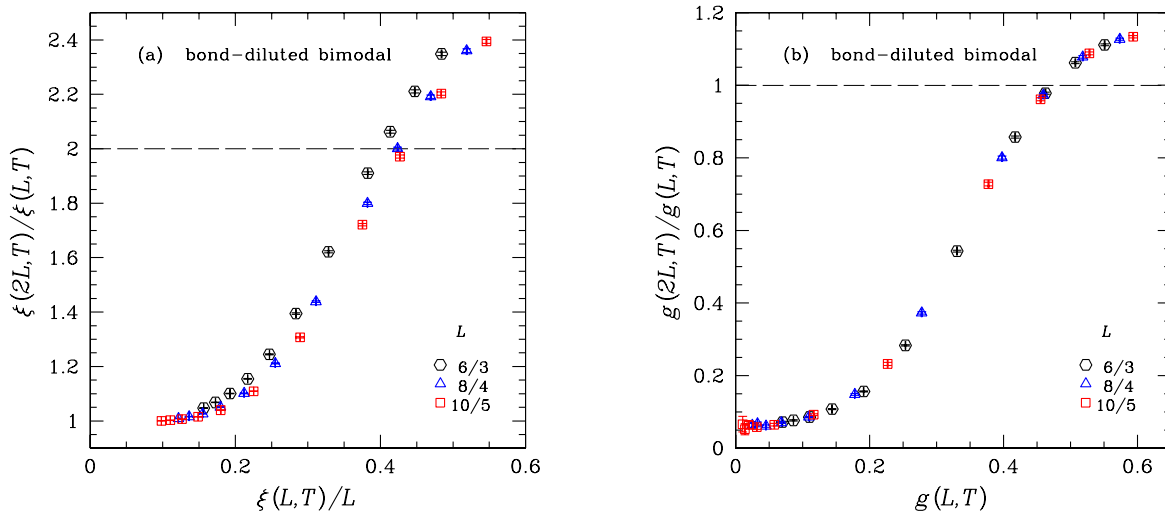
FIG. 3: (Color online) Finite-size scaling functions of the correlation length F_ξ and the Binder cumulant F_g of the four-dimensional Edwards-Anderson Ising spin glass with bond-diluted bimodal disorder for different system sizes L. The comparison between the scaling function F_ξ as a function of the finite-size correlation length $\xi(L,T)/L$ is shown in panel (a) and F_g as a function of the Binder cumulant $g(L,T)$ shown in panel (b). Given the small system sizes studied, the Binder cumulant is more suited as a scaling variable, since it displays clearly smaller scaling corrections than the finite-size correlation length. The data collapse for the scaling function F_g is excellent for the small system sizes studied. The broken lines in both panels indicate the condition that defines the effective critical point T_c^*.

TABLE II: Parameters of the simulations for the Gaussian-distributed disorder. L denotes the system size. N_{sa} is the number of samples and N_{sw} is the total number of Monte Carlo sweeps performed on a single sample for each of the $2N_T$ replicas. $T_{\min}$ and $T_{\max}$ are the lowest and highest temperatures simulated, and N_T is the number of temperatures used in the parallel tempering method.

L	N_{sa}	N_{sw}	N_T	$T_{\min}$	$T_{\max}$
3	20000	131072	29	1.400	3.061
4	20000	131072	29	1.400	3.061
5	20000	131072	29	1.400	3.061
6	20000	131072	29	1.400	3.061
8	3500	524288	29	1.400	3.061
10	2000	524288	29	1.400	3.061

TABLE III: Parameters of the simulations for the bond-diluted bimodal disorder distribution. For details see Table II.

L	N_{sa}	N_{sw}	N_T	$T_{\min}$	$T_{\max}$
3	102400	20000	11	0.950	1.800
4	107680	40000	11	0.950	1.800
5	101699	40000	11	0.950	1.800
6	101664	40000	11	0.950	1.800
8	41408	100000	21	0.950	1.800
10	24160	100000	21	0.950	1.800

Panel (b) of Fig. 1 shows the data for the Binder cumulant $g(L,T)$ as a function of the temperature for different system sizes. The data cross cleanly at the critical temperature of $T_c = 1.805(10)$ and there is no shift as for

the crossings of the finite-size correlation length [panel (a)]. In contrast to the situation in three dimensions the data splay out very well below T_c making a precise determination of T_c from the Binder cumulant data possible, although the error bars on the Binder cumulant are slightly larger than the ones on the correlation length. We consider Fig. 1 as a first indication that it might be profitable to use $g(L,T)$ instead of $\xi(L,T)/L$ as scaling variable in the finite-size scaling analysis in four dimensions, possibly because T_c is considerably larger and thus the crossing point further away from $T = 0$ where $g \to 1$.

In Table IV (Appendix C) we present the results for the critical quantities we obtain from the quotient method using T_c^* defined from the crossings of ξ/L and g, respectively. While the results for T_c, $g(T_c)$, and ν show no noticeable scaling corrections, the ones for η, $\xi(L,T_c)/L$, and in a minor extent γ indicate clearly the presence of such corrections. In conclusion we obtain the following values for the critical quantities for the undiluted Gaussian disorder:

$$
\begin{aligned}
T_c &= 1.805(10), \\
g(T_c) &= 0.470(5), \\
\nu &= 1.02(2), \\
\eta &= -0.275(25), \\
\gamma &= 2.32(8).
\end{aligned}
\tag{27}
$$

Our value for η contains a crude extrapolation to the thermodynamic limit, which makes use of the fact that corrections to scaling seem to be disappearing very fast with increasing system size. The result we give for η
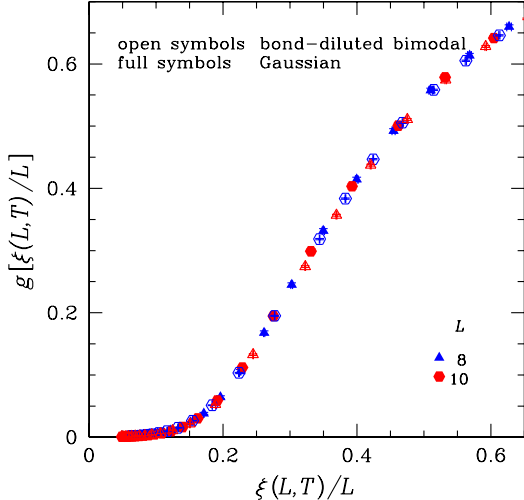
FIG. 4: (Color online) Comparison of the Binder cumulant g as a function of the finite-size correlation length for the four-dimensional EA Ising spin glass with Gaussian (full symbols) and diluted bimodal disorder (open symbols) for different system sizes $L \geq 8$. Both functions agree very well; further evidence for universal behavior.

is justified later in Fig. 6. The value for γ is determined from our estimates of ν and η. Its value depends much more on a precise determination of ν than on a precise determination of η. We did not try to determine an infinite-volume extrapolation for the value of $\xi(L, T_c)/L$, which is another universal quantity, because the data from such a limited range of system sizes do not allow for a controlled extrapolation. Note that in the case of η, where one might expect similar problems, we rely on the convergence of the estimates from opposing sides for the different scaling functions (see Fig. 6) thus allowing for a somewhat more reliable extrapolation. Taking into account the presence of corrections to scaling, the estimates for the critical exponents obtained from the extended scaling method of Campbell *et al.*[59] (see Appendix D) do agree with our final estimates.

B. Diluted bimodal disorder

Panel (a) in Fig. 2 shows data for the finite-size correlation length as a function of the temperature for different system sizes. With increasing system size L the data show—as for the Gaussian data—a noticeable shift of the effective critical temperature T_c toward a smaller value. Taking the data for $L \geq 6$ and neglecting the remaining scaling corrections we obtain $T_c = 1.042(5)$. Panel (b) in Fig. 2 shows the data for the Binder cumulant $g(L, T)$ as a function of the temperature for different system sizes. The data cross cleanly, and we determine $T_c = 1.0385(25)$. We find again as in the Gaussian case that the Binder cumulant has smaller corrections to scal-

ing than the correlation length. This is confirmed by a comparative analysis of the finite-size scaling functions F_ξ as a function of $\xi(L, T)/L$ and F_g as a function of $g(L, T)$.

The finite-size scaling functions F_ξ and F_g contain in principle the very same information. However, due to the fact that we are working in a restricted range of lattice sizes there are visible differences. In panel (a) of Fig. 3 we show the scaling function F_ξ and in panel (b) of Fig. 3 of the corresponding scaling function F_g. Comparing the two figures one clearly sees that F_g shows much smaller finite-size corrections than F_ξ. This figure is clear evidence that for our data the Binder cumulant $g(L, T)$ is better suited as a scaling variable than the correlation length $\xi(L, T)/L$. This is in contrast to recent work in three dimensions, where typically $\xi(L, T)/L$ is used instead of $g(L, T)$ in order to cleanly determine T_c and the critical exponents.[3,34] The reason for this behavior is possibly given by the following arguments: First, in three dimensions the crossing of the Binder cumulant data is at a rather flat angle making it difficult to determine T_c precisely, while in four dimensions the crossing is at a much steeper angle. Second, the range of available system sizes in four dimensions is smaller than in three dimensions and therefore the use of observables with small scaling violations is better in order to have good control over the estimates of the critical quantities.

In Table IV (Appendix C) we present the results for the critical quantities we have obtained from the quotient method using T_c^*, defined either from the crossings of ξ/L or g, respectively. We obtain the following values for the critical quantities for the link-diluted bimodal disorder distribution:

$$\begin{aligned}
T_c &= 1.0385(25), \\
g(T_c) &= 0.472(2), \\
\nu &= 1.025(15), \\
\eta &= -0.275(25), \\
\gamma &= 2.33(6).
\end{aligned} \tag{28}$$

The estimate of η is justified below in Fig. 6.

C. Comparison of finite-size scaling functions

The results for the critical quantities given in Eqs. (27) and (28) are consistent with universal critical behavior of the four-dimensional Edwards-Anderson Ising spin-glass model. We further strengthen this result by comparing the finite-size scaling functions of the two models not only at the critical point but within the whole scaling region. The direct comparison of the finite-size scaling functions has shown to be probably the best approach to check for universality in spin glasses as it allows for a completely parameter-free comparison of different models (see Refs. 2 and 3 for a comparison of different models in three space dimensions). In Fig. 4 we compare
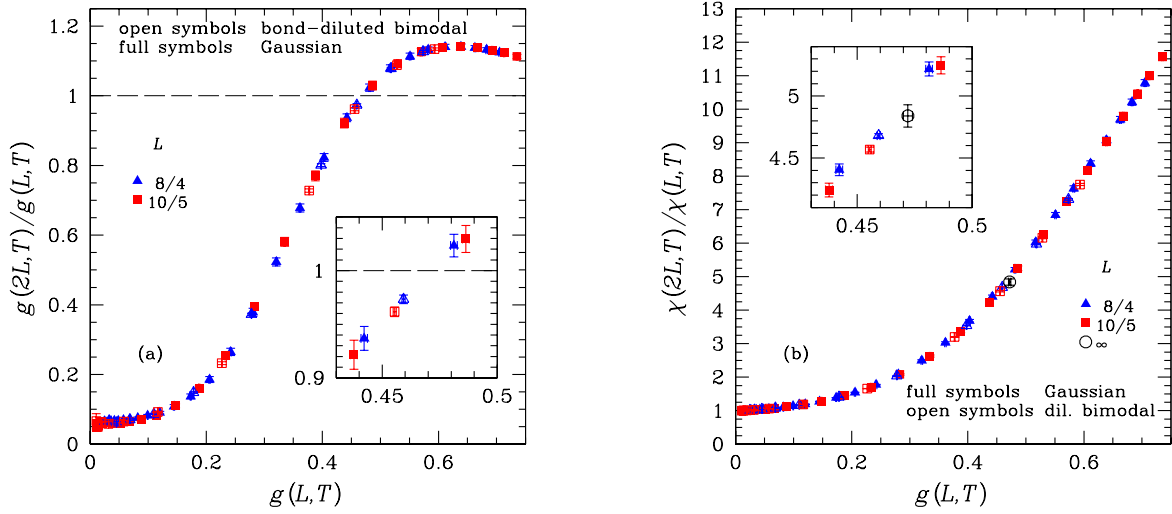
FIG. 5: (Color online) Comparison of the finite-size scaling function F_g shown in panel (a) and F_χ shown in panel (b) for the four-dimensional model with Gaussian and bond-diluted bimodal disorder for different system sizes L as a function of the Binder cumulant g. The data for both finite-size scaling functions show very little scaling corrections. The fact that the curves for the two different models for F_g and F_χ fall on one single curve is a strong evidence for universal critical behavior of the four-dimensional Edwards-Anderson model. The insets in both panels present an enlarged view around the critical point [which is at $g(L, T_c) = 0.472(2)$] and show in more detail that the data for the finite-size scaling functions collapse onto one single master curve. The data point with the label "∞" in panel (b) indicates our infinite-volume extrapolation of F_χ at criticality corresponding to $\eta = 0.275(25)$ and $g(T_c) = 0.472(2)$.
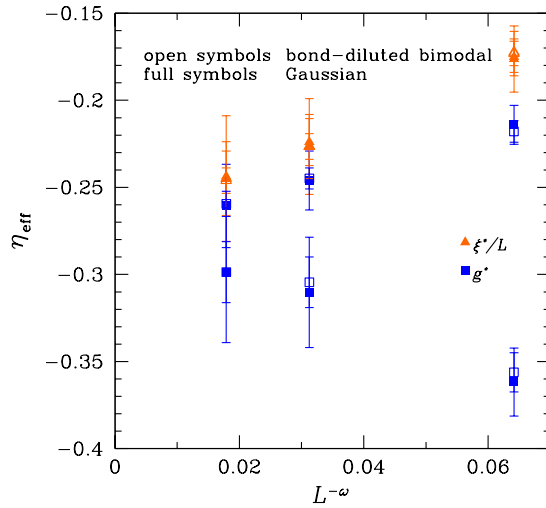

FIG. 6: (Color online) Convergence of the effective exponent η_{eff} defined from F_χ and F_ζ through Eqs. (25) and (26), respectively, at T_c^* (defined by ξ^*/L and g^*) as a function of $L^{-\omega}$ with $\omega = 2.5$. The data for the Gaussian (full symbols) and bond-diluted bimodal (open symbols) disorder are consistent within errors and they are also consistent with a unique value of $\eta = -0.275(25)$ for an infinite-volume extrapolation.

the Binder cumulant $g(L, T)$ plotted against the correlation length $\xi(L, T)/L$ for the two different disorder distributions. The data agree and collapse onto a single

master curve, which is a strong evidence for universality. In panel (a) of Fig. 5 we compare the finite-size scaling function F_g as a function of the Binder cumulant $g(L, T)$. The data collapse again within error bars onto a single master curve. This result illustrates once more that the estimates of the critical exponent ν of the two disorder distributions do agree within error bars. In panel (b) of Fig. 5 we compare the finite-size scaling function F_χ as a function of the Binder cumulant $g(L, T)$. Again the data collapse is within the error bars, which means that also the estimates of the critical exponent η of the two models coincide within errors.

Finally, in Fig. 6 we show the convergence of the effective exponent η_{eff} from F_χ and F_ζ through Eqs. (25) and (26), respectively, measured at the crossings of ξ/L and g as a function of $L^{-\omega}$. All the scaling corrections are compatible with a leading correction-to-scaling exponent of $\omega \approx 2.5$. Clearly, this is only an effective exponent because for the small system sizes we have at hand, we cannot expect to be in the asymptotic scaling regime where the leading correction-to-scaling exponent dominates. However, our estimate of ω is consistent with the assumption that the corrections to scaling we see are dominated by the first nonleading correction-to-scaling term since our value for ω is roughly twice as large as the one obtained from high-temperature expansion studies in Ref. 5 or alternatively they might be due to remaining analytic corrections with an effective correction-to-scaling exponent given by $2 - \eta$. Figure 6 shows that an infinite-volume extrapolation of the different estimates is
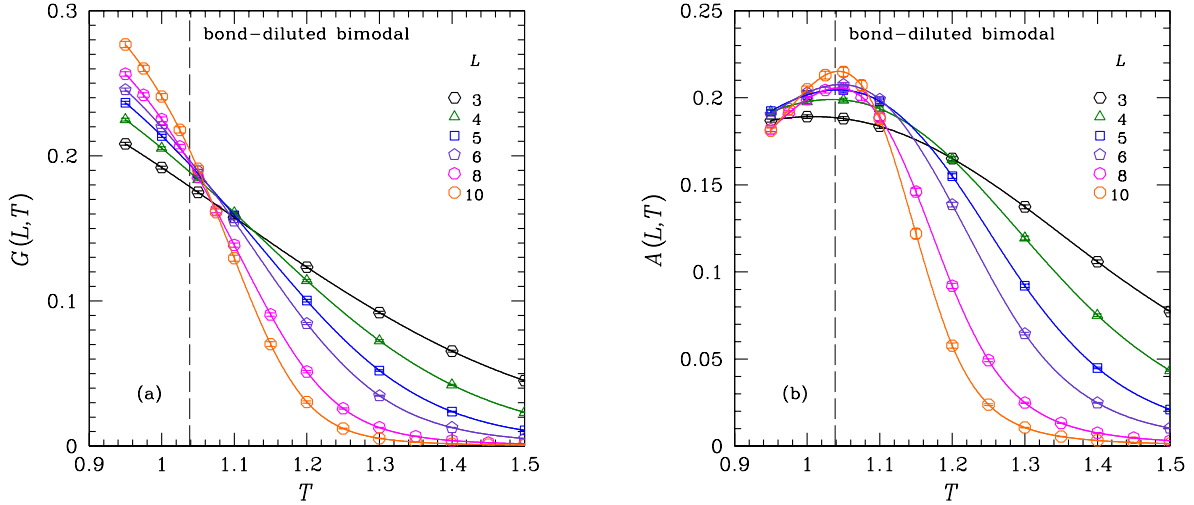
FIG. 7: (Color online) The Guerra parameter $G(L,T)$ [panel (a)] and the lack of self-averaging parameter $A(L,T)$ [panel (b)] as a function of temperature T for the four-dimensional Edwards-Anderson model with bond-diluted bimodal disorder for different system sizes L. The crossings shift noticeably to a smaller effective T_c in both cases for increasing system size making an accurate determination of T_c from these data difficult and imprecise. In the case of $A(L,T)$ the fact that the crossing occurs close to the maximum of the curves makes the situation for an accurate determination of T_c (vertical dashed lines) impossible.

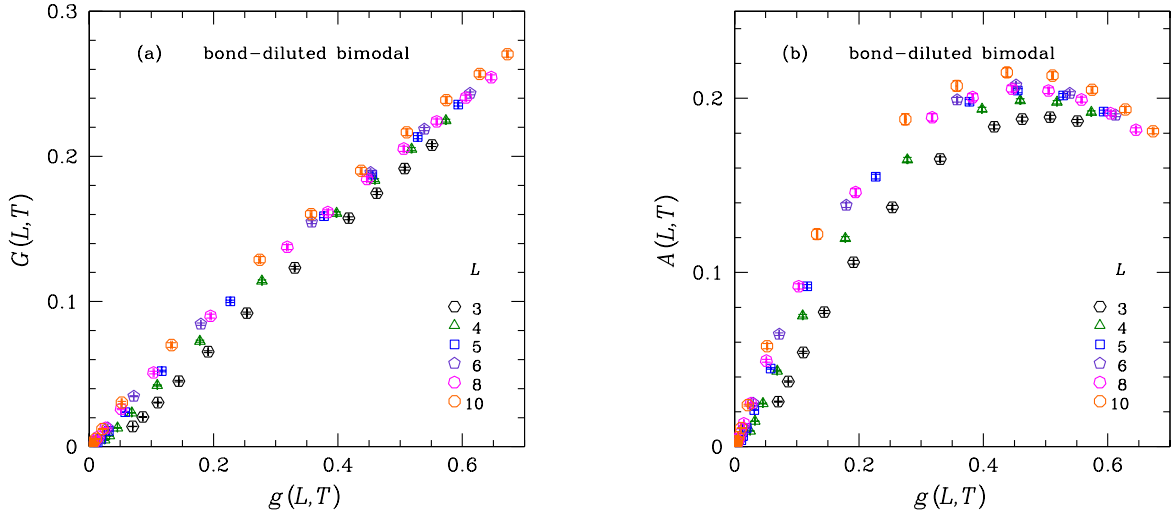FIG. 8: (Color online) The Guerra parameter $G(L,T)$ [panel (a)] and the lack of self-averaging parameter $A(L,T)$ [panel (b)] as a function of the Binder cumulant $g(L,T)$ for the four-dimensional Edwards-Anderson Ising spin glass with bond-diluted bimodal disorder for different system sizes L. Both $G(L,T)$ and $A(L,T)$ display large corrections to scaling.

compatible with a unique value of $\eta = -0.275(25)$ in the infinite-volume limit for *both* models.

IV. CONCLUSIONS

We have tested universality in four-dimensional Ising spin glasses and computed precise estimates of the critical parameters of the model with both Gaussian and link-diluted bimodal disorder. Our results show that the different critical exponents for different disorder distri-

butions agree well. Furthermore, by plotting the Binder ratio as a function of the correlation length, we show that four-dimensional Ising spin glasses (with compact disorder distributions) seem to share the same universality class. Furthermore, we compute different finite-size scaling functions in four space dimensions defined via ratios of different observables and show that these show small corrections to scaling, especially when studied as a function of the Binder parameter. The results presented thus indicate that universality is not violated in four-dimensional spin glasses.
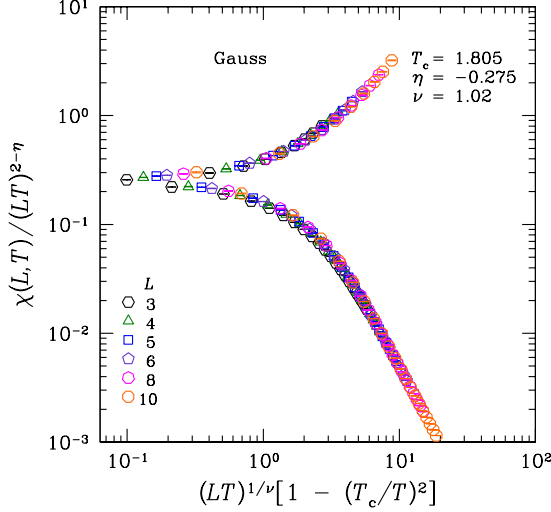
FIG. 9: (Color online) Scaling plot of the spin-glass susceptibility according to the extended scaling approach for the system with Gaussian disorder. We use the estimates of the critical exponents presented in Eqs. (27). The data for the susceptibility scale very well with our estimates of the critical parameters.

APPENDIX A: OTHER OBSERVABLES

In this appendix we discuss the quantities that have been shown to be less useful in the study of the location of the spin-glass transition and the issue of the universality of the four-dimensional Edwards-Anderson model. We have decided to present our results concerning these quantities since our data for the bond-diluted bimodal coupling distribution has by far the best statistics in the context of Monte Carlo simulations of the four-dimensional Edwards-Anderson model.

Panel (a) of Fig. 7 shows the data for the Guerra parameter $G(L,T)$ defined in Eq. (15) as a function of the temperature for different system sizes. The cross-

ings of the data for increasing system sizes shift noticeably toward smaller temperatures. We find that $G(L,T)$ has rather large finite-size corrections and also relatively large errors compared to, e.g., the Binder cumulant and therefore is not well suited for an accurate determination of T_c. The same conclusions have been found by Ballesteros *et al.*[34] for the three-dimensional bimodal Ising spin glass. Note, however, that in the case of mean-field spin glasses the situation may be different as the Guerra parameter has shown to be more efficient in locating a spin-glass transition than the Binder cumulant in certain situations.[58]

Panel (b) of Fig. 7 shows the data for the lack of self-averaging parameter $A(L,T)$ defined in Eq. (14) as a function of the temperature for different system sizes. This parameter is related to the Guerra parameter and also shows large finite-size corrections and due to the fact that the crossings happen close to the maximum of the curves, where the slope changes very fast, the crossings (apart from the fact that they move noticeably) cannot be determined reliably. This fact together with the rather large relative error makes that this quantity the least suited for a precise determination of the critical temperature of all the quantities discussed here.

In Fig. 8 we show the data for $G(L,T)$ [panel (a)] and $A(L,T)$ [panel (b)] as a function of the Binder cumulant. Clearly, the two quantities have strong finite-size corrections. The fact that these quantities may present such strong finite-size scaling corrections should be kept in mind when, e.g., the behavior of $A(L,T)$ is used to make statements on the nature of the spin-glass phase as it is done in Ref. 58.

APPENDIX B: DERIVATION OF THE QUOTIENT RELATIONS

In the following, we derive Eqs. (22) and (23) in detail. Starting from the definition of F_ξ, we can write,

$$\xi(sL,T) = F_\xi[\xi(L,T)/L;s]\xi(L,T). \qquad (B1)$$

Taking the derivative with respect to T, using the chain rule and finally dividing by $\partial_T\xi(L,T)$, we arrive at

$$\frac{\partial_T\xi(sL,T)}{\partial_T\xi(L,T)} = \frac{\xi(L,T)}{L}\partial_x F_\xi(x;s) + F_\xi[\xi(L,T)/L]. \quad (B2)$$

Note that

$$F_{\partial_T\xi}[\xi(L,T)/L;s] = \frac{\partial_T\xi(sL,T)}{\partial_T\xi(L,T)}. \qquad (B3)$$

Using the fact that at the effective critical point, where $\xi(L,T_c^*)/L = x^*$, we have $F_\xi(x^*;s) = s$ and the fact that the correlation length ξ close to the critical point has a simple scaling form given in Eq. (9), we see that

$$F_{\partial_T\xi}(\xi(L,T)/L;s) = s^{1/\nu+1} + O(L^\omega)$$
$$= x^*\partial_x F_\xi(x;s)\big|_{x=x^*} + s, \qquad (B4)$$

TABLE IV: Results from the quotient method for the Gaussian (top) and link-diluted bimodal (bottom) disorder at T_c^* determined from the crossings of ξ/L and g, respectively. We group the data according to the observables which were used to compute them (separated by vertical bars).

Crossings of ξ/L	L	T_c^*	ξ^*/L	$\partial_\xi F_\xi^*$	ν_{eff}	F_χ^*	η_{eff}	γ_{eff}	F_ζ^*	η_{eff}	γ_{eff}
	3/6	1.858(6)	0.396(2)	5.00(3)	1.007(8)	4.52(6)	−0.18(2)	2.20(3)	1.129(7)	−0.175(9)	2.19(3)
	4/8	1.824(7)	0.418(3)	4.67(5)	1.018(14)	4.68(9)	−0.23(3)	2.27(6)	1.168(11)	−0.224(16)	2.26(5)
	5/10	1.814(8)	0.427(4)	4.62(6)	1.010(17)	4.74(12)	−0.24(4)	2.26(8)	1.184(12)	−0.244(15)	2.27(5)
Crossings of g	L	T_c^*	g^*	$\partial_g F_g^*$	ν_{eff}	F_χ^*	η_{eff}	γ_{eff}	F_ζ^*	η_{eff}	γ_{eff}
	3/6	1.803(6)	0.470(3)	2.01(5)	1.042(25)	5.14(7)	−0.36(2)	2.46(8)	1.160(9)	−0.214(12)	2.31(7)
	4/8	1.805(8)	0.469(5)	2.08(9)	1.018(42)	4.96(11)	−0.31(3)	2.35(13)	1.186(14)	−0.246(17)	2.29(11)
	5/10	1.805(8)	0.471(7)	2.06(11)	1.022(54)	4.92(14)	−0.30(4)	2.35(17)	1.198(20)	−0.261(24)	2.31(15)

Crossings of ξ/L	L	T_c^*	ξ^*/L	$\partial_\xi F_\xi^*$	ν_{eff}	F_χ^*	η_{eff}	γ_{eff}	F_ζ^*	η_{eff}	γ_{eff}
	3/6	1.0716(15)	0.400(2)	4.86(2)	1.021(7)	4.51(4)	−0.17(1)	2.22(3)	1.127(6)	−0.172(8)	2.22(3)
	4/8	1.0502(16)	0.424(2)	4.56(2)	1.025(7)	4.68(6)	−0.23(2)	2.29(3)	1.170(6)	−0.227(8)	2.28(3)
	5/10	1.0441(18)	0.433(2)	4.42(3)	1.032(9)	4.74(7)	−0.25(2)	2.32(4)	1.186(6)	−0.246(8)	2.32(3)
Crossings of g	L	T_c^*	g^*	$\partial_g F_g^*$	ν_{eff}	F_χ^*	η_{eff}	γ_{eff}	F_ζ^*	η_{eff}	γ_{eff}
	3/6	1.0390(18)	0.472(2)	2.09(3)	1.010(14)	5.12(4)	−0.36(1)	2.38(3)	1.163(5)	−0.224(7)	2.25(3)
	4/8	1.0390(18)	0.472(2)	2.06(3)	1.021(15)	4.94(5)	−0.31(2)	2.36(5)	1.185(5)	−0.245(7)	2.29(4)
	5/10	1.0384(20)	0.473(3)	2.06(4)	1.019(20)	4.92(6)	−0.30(2)	2.34(7)	1.197(6)	−0.259(8)	2.30(6)

from which we obtain Eq. (22) by dividing by s. The derivation of Eq. (23) has as the starting point

$$g(sL,T) = F_g(g(L,T);s)g(L,T), \qquad \text{(B5)}$$

and for the rest is analogous to the one given for Eq. (22).

APPENDIX C: RESULTS FROM THE QUOTIENT METHOD

In this section we list the detailed results from the quotient method. The data are grouped by the observable used to compute the estimates in Table IV.

APPENDIX D: EXTENDED SCALING

Recently, Campbell et al.[59] suggested an extended scaling approach, which allows one to extend the scaling region from $|L^{1/\nu}(T-T_c)| \lesssim 1$ to virtually $T \to \infty$. The method has the advantage in that it drastically reduces the corrections to scaling commonly found when performing a simple finite-size scaling analysis of the spin-glass susceptibility.[3] In that scaling approach the scaling equation for the susceptibility [see Eq. (12)] is modified in the following way:[60]

$$\chi_{\mathrm{SG}}(L,T) \sim (LT)^{2-\eta}\, \tilde{\chi}[B(LT)^{1/\nu}|1 - (T_c/T)^2|]. \quad \text{(D1)}$$

In Fig. 9 we illustrate the quality of the critical parameters by performing an extended finite-size scaling plot of the susceptibility for the EA spin glass with Gaussian disorder obtained from the quotient method.[34,46] Similar results are obtained for the model with bond-diluted bimodal disorder, as well as other observables.

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